

AN ARITHMETICAL EXCURSION VIA STONEHAM NUMBERS

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To Professor Peter Borwein on his 60th birthday

ABSTRACT. Let p be a prime and b a primitive root of p^2 . In this paper, we give an explicit formula for the number of times a value in $\{0, 1, \dots, b-1\}$ occurs in the periodic part of the base b expansion of $1/p^m$. As a consequence of this result, we prove two recent conjectures of Francisco Aragón, David Bailey, Jonathan Borwein, and Peter Borwein concerning the base b expansion of Stoneham numbers.

1. INTRODUCTION

Let $b \geq 2$ be an integer. A real number $\alpha \in (0, 1)$ is called *b-normal* if in the base b expansion of α the asymptotic frequency of the occurrence of any word $w \in \{0, 1, \dots, b-1\}^*$ of length n is $1/b^n$. A canonical example of a such a number is Champernowne's number,

$$C_{10} := 0.123456789101112131415161718192021 \dots,$$

which given here in base 10, is the size-ordered concatenation of \mathbb{N} (each number written in base 10) with a radix point out front. Champernowne's number was shown to be 10-normal by Champernowne [5] in 1933 and transcendental by Mahler [6] in 1937. In 1973, Stoneham [9] defined the following class of numbers.

Let $b, c \geq 2$ be relatively prime integers. The *Stoneham number* $\alpha_{b,c}$ is given by

$$\alpha_{b,c} := \sum_{n \geq 1} \frac{1}{c^n b^{c^n}}.$$

Stoneham [9] showed that $\alpha_{2,3}$ is 2-normal. A new proof of this result was given by Bailey and Misiurewicz [4] and finally in 2002, Bailey and Crandall [3] proved that $\alpha_{b,c}$ is b -normal for all coprime integers $b, c \geq 2$; see also Bailey and Borwein [2]. Transcendence of $\alpha_{b,c}$ follows by Mahler's method.

Recently Francisco Aragón, David Bailey, Jonathan Borwein, and Peter Borwein [1] made the following conjectures concerning properties of the base-4 expansion of the Stoneham number $\alpha_{2,3}$ and the base-3 expansion of $\alpha_{3,5}$ (here we have fixed a few small typos in their published conjectures).

Conjecture 1.1 (Aragón, Bailey, Borwein, and Borwein [1]). *Denote the base-4 expansion of $\alpha_{2,3}$ by $\alpha_{2,3} := \sum_{k \geq 1} a_k 4^{-k}$, with $a_k \in \{0, 1, 2, 3\}$. Then for all $n \geq 0$ one has*

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- (i) $\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} \left(e^{\frac{\pi i}{2}}\right)^{a_k} = - \begin{cases} i, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even,} \end{cases}$
- (ii) $a_k = a_{3^n+k} = a_{2 \cdot 3^n+k}$ for $k = \frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+1, \dots, \frac{3}{2}(3^n+1)+3^n-1$.

Conjecture 1.2 (Aragón, Bailey, Borwein, and Borwein [1]). *Denote the base-3 expansion of $\alpha_{3,5}$ by $\alpha_{3,5} := \sum_{k \geq 1} a_k 3^{-k}$, with $a_k \in \{0, 1, 2\}$. Then for all $n \geq 0$ one has*

- (i) $\sum_{k=1+5^{n+1}}^{1+5^{n+1}+4 \cdot 5^n} \left(e^{\frac{\pi i}{3}}\right)^{a_k} = (-1)^n e^{\frac{i\pi}{3}}$
- (ii) $a_k = a_{4 \cdot 5^n+k} = a_{8 \cdot 5^n+k} = a_{12 \cdot 5^n+k} = a_{16 \cdot 5^n+k}$ for $k = 5^{n+1} + j$, with $j = 1, \dots, 4 \cdot 5^n$.

We note here that the Stoneham numbers $\alpha_{b,c}$ are in some ways very similar to Champernowne's numbers. They are not concatenations of consecutive integers, but the concatenation of periods of certain rational numbers. The Stoneham numbers share arithmetic properties with the following construction. Let $b, c \geq 2$ be coprime integers and let w_n be the word $w \in \{0, 1, \dots, b-1\}^*$ of minimal length such that

$$\left(\frac{1}{c^n}\right)_b = 0.\overline{w_n},$$

where $(x)_b$ denotes the base b expansion of the real number x and \overline{w} denotes the infinitely repeated word w . Then the Stoneham numbers are, morally at least, arithmetically similar to the numbers

$$0.w_1 w_2 w_3 w_4 w_5 \cdots w_n \cdots,$$

which are given by concatenating the words w_n .

Remark 1.3. While we will be considering the base 4 expansion of $\alpha_{2,3}$ we are still dealing with a normal number; $\alpha_{2,3}$ is also 4-normal. This is given by a result of Schmidt [8] who proved in 1960 that the r -normal real number x is s -normal if and only if $\log r / \log s \in \mathbb{Q}$.

2. BASE b EXPANSIONS OF RATIONALS

To prove the above conjectures in as much generality as possible we will need to consider how we write a reduced fraction a/k in base b . Such an algorithm is well-known, but we remind the reader here, as it will be useful to have the general framework for the proof of Conjectures 1.1 and 1.2. To write a/k in base b , we use a sort of modified division algorithm; see Figure 1.

We record here some relationships between the parts of the base b algorithm which have interesting consequences.

Lemma 2.1. *Suppose $b, k \geq 2$ are coprime, and that r_j and q_j are defined by the base b algorithm for a/k . Then $\gcd(r_i, k) = 1$.*

Proof. Suppose that $p|k$. By induction on i . Firstly, $r_0 = a$ and by assumption $\gcd(r_0, k) = \gcd(a, k) = 1$.

Now suppose that $\gcd(r_i, k) = 1$, so that also $\gcd(r_i b, k) = 1$. Then

$$r_{i+1} = r_i b - q_{i+1} k \equiv r_i b \not\equiv 0 \pmod{p},$$

since $\gcd(b, k) = 1$. Thus $\gcd(r_{i+1}, k) = 1$. □

Base b Algorithm for a/k .

Let $b, k \geq 2$ be integers and $a \geq 1$ be an integer coprime to k . Set $r_0 = a$ and write

$$\begin{aligned} r_0 b &= q_1 k + r_1 \\ r_1 b &= q_2 k + r_2 \\ &\vdots \\ r_{j-1} b &= q_j k + r_j \\ &\vdots \end{aligned}$$

where $q_j \in \{0, 1, \dots, b-1\}$ and $r_j \in \{0, 1, \dots, k-1\}$ for each j . Stop when $r_n = r_0$. Then

$$\left(\frac{a}{k}\right)_b = 0.\overline{q_1 q_2 \dots q_n}.$$

FIGURE 1. The base b algorithm for the reduced rational a/k .

Also, we have that equivalent r_j give equal q_j .

Lemma 2.2. *Suppose $b, k \geq 2$ are coprime, and that r_j and q_j are defined by the base b algorithm for the reduced fraction a/k . We have $r_i \equiv r_j \pmod{b}$ if and only if $q_i = q_j$.*

Proof. Suppose that $r_i \equiv r_j \pmod{b}$. Then subtracting the equations $r_{i-1}b = q_i k + r_i$ and $r_{j-1}b = q_j k + r_j$ and taking the result modulo b , we get that $b|(q_i - q_j)k$ so that since $\gcd(b, k) = 1$ gives $b|(q_i - q_j)$. Since $q_i, q_j \in \{0, 1, \dots, b-1\}$, we thus have that $q_i = q_j$.

Conversely, suppose that $q_i = q_j$. Then subtracting the defining equations for q_i and q_j and taking the result modulo b gives the desired result. \square

Indeed, the value of q_j can be determined by the residue class of r_j modulo b and the value of k^{-1} modulo b .

Lemma 2.3. *Suppose $b, k \geq 2$ are coprime, and that r_j and q_j are defined by the base b algorithm for the reduced fraction a/k . We have $r_i \equiv j \pmod{b}$ if and only if $q_i = (-jk^{-1} \pmod{b})$, where the value of $-jk^{-1}$ modulo b is taken in the interval $[0, b-1]$.*

Proof. If $r_i \equiv j \pmod{b}$, then the equation $r_{i-1}b = q_i k + r_i$ gives $q_i k \equiv -j \pmod{b}$, which in turn gives that $q_i \equiv -jk^{-1} \pmod{b}$. Since $q_i \in [0, b-1]$ we are done with this direction of proof.

Conversely, suppose that $q_i = (-jk^{-1} \pmod{b})$. Then surely, $q_i \equiv -jk^{-1} \pmod{b}$ and so $q_i k \equiv -j \pmod{b}$. Thus, again using $r_{i-1}b = q_i k + r_i$, we have that $r_i \equiv j \pmod{b}$. \square

The following Lemma is a direct corollary of Lemma 2.3.

Lemma 2.4. *Suppose $b, k \geq 2$ are coprime, and that r_j and q_j are defined by the base b algorithm for the reduced fraction a/k . We have $r_i \equiv 0 \pmod{b}$ if and only if $q_i = 0$.*

Proof. Apply Lemma 2.3 with $j = 0$. \square

We will use the following classical theorem (see [7, Theorem 12.4]) and lemma.

Theorem 2.5. *Let b be a positive integer. Then the base b expansion of a rational number either terminates or is periodic. Further, if $r, s \in \mathbb{Z}$ with $0 < r/s < 1$ where $\gcd(r, s) = 1$ and $s = TU$, where every prime factor of T divides b and $\gcd(U, b) = 1$, then the period length of the base b expansion of r/s is the order of b modulo U , and the preperiod length is N where N is the smallest positive integer such that $T|b^N$.*

Theorem 2.5 tells us that the base b expansion of a/k is purely periodic (recall for us $\gcd(b, k) = 1$), and that the minimal period is $\text{ord}_k b$, which divides $\varphi(k)$ so that this also is a period. This result can be exploited using the following number-theoretic result.

Lemma 2.6. *Any primitive root a of p^2 is also a primitive root of p^k for any integer $k \geq 2$.*

Applying the previous lemma gives the following result.

Lemma 2.7. *Let $0 < a/p^m < 1$ be a rational number in lowest terms and let $b \geq 2$ be an integer that is a primitive root of p^2 . Suppose that $(1/p^m)_b = \overline{q_1 q_2 \cdots q_n}$ is given by the base b algorithm. Then*

$$\left(\frac{a}{p^m}\right)_b = \overline{q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)}}$$

where σ is a cyclic shift on n letters.

Proof. This is a direct consequence of the base b algorithm. \square

As a consequence of the above lemmas we are able to provide the following characterisation of certain base b expansions.

Proposition 2.8. *Let $n \geq 1$ be an integer, p be an odd prime, $b \geq 2$ be an integer coprime to p , and q_j and r_j be given by the base b algorithm for the reduced fraction a/p^m . If b is a primitive root of p^2 , then $\text{period}(a/p^m) = \varphi(p^m)$ and*

$$\#\{j \leq \varphi(p^m) : q_j = 0\} = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor.$$

Proof. The fact that $\text{period}(a/p^m)_b = \varphi(p^m)$ follows directly from b being a primitive root of p^2 , Lemma 2.6 and Theorem 2.5. This further implies that the $\varphi(p^m)$ r_i s given by the base b algorithm for a/p^m are distinct. Applying Lemma 2.1 gives that

$$(1) \quad \{r_1, r_2, \dots, r_{\varphi(p^m)}\} = \{i \leq p^m : \gcd(i, p) = 1\}.$$

Also recall that

$$\left(\frac{a}{p^m}\right)_b = \overline{q_1 q_2 \cdots q_{\varphi(p^m)}},$$

and that by Lemma 2.4, $q_i = 0$ if and only if $r_i \equiv 0 \pmod{b}$. Note that there are exactly

$$\left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^m}{bp} \right\rfloor = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor$$

elements of $\{i \leq p^m : \gcd(i, p) = 1\}$ which are divisible by b . Thus using the set equality (1), we have that there are exactly $\lfloor p^m/b \rfloor - \lfloor p^{m-1}/b \rfloor$ elements of $\{r_1, r_2, \dots, r_{\varphi(p^m)}\}$ divisible by b . Appealing to Lemma 2.4 we then have that there are $\lfloor p^m/b \rfloor - \lfloor p^{m-1}/b \rfloor$ of $q_1, q_2, \dots, q_{\varphi(p^m)}$ such that $q_j = 0$. \square

Note that while we record the $q_i = 0$ case because of its simplicity, the method can be applied to count any value of q_i that is desired by using the appropriate case of Lemma 2.3. In fact, we will do this in a few special cases to prove Conjectures 1.1 and 1.2.

3. THE BASE b EXPANSION OF THE STONEHAM NUMBER $\alpha_{b,p}$

We will need properties for both the base b expansion and the base b^2 expansions of the Stoneham number $\alpha_{b,p}$.

Proposition 3.1. *Let $b, p \geq 2$ be coprime integers with p a prime. Denote the b -ary expansion of $\alpha_{b,p}$ as*

$$\alpha_{b,p} = \sum_{j \geq 1} \frac{1}{p^j b^{p^j}} = \sum_{k \geq 1} \frac{a_k}{b^k},$$

where $a_k \in \{0, 1, \dots, b-1\}$ and write

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m} \right)_b = .q_1 q_2 \cdots q_n$$

where the q_i s are determined by the base b algorithm, so $n = \text{ord}_{p^m} b$. Then $q_i = a_{p^m+jn+i}$ for each $i \in \{1, 2, \dots, n\}$ and each $j \in \{0, 1, 2, \dots, \frac{p \cdot \varphi(p^m)}{\text{ord}_{p^m} b} - 1\}$.

Lemma 3.2. *Let $b, c \geq 2$ be coprime. Then for any $m \geq 1$ we have*

$$\left| \alpha_{b,c} - \sum_{n=1}^m \frac{1}{c^n b^{c^n}} \right| < \frac{1}{b^{c^{m+1}}}.$$

That is, the b -ary expansion of $\alpha_{b,c}$ agrees with the b -ary expansion of its m -th partial sum up to the (c^{m+1}) -th place.

Proof. Let $m \geq 1$ and note that

$$\sum_{n \geq m+1} \frac{1}{c^n} = \frac{1}{c^{m+1} - c^m} < 1.$$

Using this fact, we have that

$$\left| \alpha_{b,c} - \sum_{n=1}^m \frac{1}{c^n b^{c^n}} \right| = \sum_{n \geq m+1} \frac{1}{c^n b^{c^n}} < \frac{1}{b^{c^{m+1}}} \sum_{n \geq m+1} \frac{1}{c^n} < \frac{1}{b^{c^{m+1}}},$$

which is the desired result. \square

Proof of Proposition 3.1. Let $m \geq 1$, $s_m = p^m b^{p^m}$, and define the positive integer r_m by

$$\frac{r_m}{s_m} := \sum_{n=1}^m \frac{1}{p^n b^{p^n}}.$$

We have then that

$$\gcd(r_m, s_m) = \gcd(r_m, p^m b^{p^m}) = \gcd(r_m, pb) = 1.$$

We apply Theorem 2.5 with $b = b$, $r = r_m$, $s = s_m$, $T = b^{p^m}$, and $U = p^m$ to give that the period length of the base b expansion of r_m/s_m is the order of b modulo p^m , which we will write

$$\text{period}(r_m/s_m) = \text{ord}_{p^m} b,$$

and the preperiod length of r_m/s_m is p^m , which we will write

$$\text{preperiod}(r_m/s_m) = p^m.$$

Combining the observations of the previous paragraph with Lemma 3.2, gives that

$$(2) \quad a_{p^m+1} a_{p^m+2} \dots a_{p^{m+1}} = \underbrace{www \dots w}_{\frac{p \cdot \varphi(p^m)}{\text{ord}_{p^m} b} \text{ times}},$$

where $w = q_1 q_2 \dots q_{\text{ord}_{p^m} b}$ is a word on the alphabet $\{0, 1, \dots, b\}$ with length $\text{ord}_{p^m} b$. To finish the proof of this Proposition, it is enough to realise that

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m} \right)_b = .\overline{w},$$

where w is as defined in the previous sentence, which follows directly from the definition of $\alpha_{b,p}$. This proves the theorem. \square

Proposition 3.3. *Let $b, p \geq 2$ be coprime integers with p a prime. Denote the b^2 -ary expansion of $\alpha_{b,p}$ as*

$$\alpha_{b,p} = \sum_{j \geq 1} \frac{1}{p^j b^{p^j}} = \sum_{k \geq 1} \frac{a_k}{b^{2k}},$$

where $a_k \in \{0, 1, \dots, b^2 - 1\}$ and write

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m} \right)_{b^2} = .\overline{q_1 q_2 \dots q_n}$$

where the q_i s are determined by the base b^2 algorithm, so $n = \text{ord}_{p^m} b^2$. Then $q_i = a_{\frac{p}{2}(p^{m-1}+1)+jn+i}$ for each $i \in \{0, 1, \dots, n-1\}$ and each $j \in \{0, 1, 2, \dots, \frac{p \cdot \varphi(p^m)}{\text{ord}_{p^m} b} - 1\}$.

Proof. This proof follows by the same method as the previous proof using the identity

$$\frac{1}{p^n b^{p^n}} = \frac{b^p}{p^n} \cdot \frac{1}{(b^2)^{\frac{p}{2}(p^{n-1}+1)}},$$

and the fact that $\text{ord}_{p^m} b^2 = \frac{\text{ord}_{p^m} b}{b}$. As the proof follows closely that of Proposition 3.1, the details are left to the reader. \square

It is worth noting that Propositions 3.1 and 3.3 are full generalisations of Conjectures 1.1(ii) and 1.2(ii), respectively.

4. THE ARAGON, BORWEIN, BORWEIN, AND BAILEY CONJECTURES

In this section, we apply the results of Section 3 to prove Conjectures 1.1 and 1.2. As it turns out, the proof of Conjecture 1.2 is a bit more straightforward, so we present its proof first.

Proof of Conjecture 1.2. For convenience let us write $\omega := e^{i\pi/3}$ and let r_i and q_i be given by the base b algorithm for $1/5^n$. Note that by Proposition 3.1, we have that

$$\sum_{k=2+5^{n+1}}^{2+5^{n+1}+4 \cdot 5^n} \omega^{a_k} = \sum_{j=0}^2 \#\{i \leq \varphi(5^{n+1}) : q_i = j\} \cdot \omega^j.$$

Now $\#\{i \leq \varphi(5^n) : q_i = j\}$ can be given by looking at where the number 5^n lies modulo 15. Since for every 15 consecutive numbers 12 of them are coprime to 5 and these 12 fall into the 3 equivalence classes modulo 3 with an equal frequency of 4 times each, we need only look at the remainder of 5^n modulo 15. An easy calculation gives that

$$5^n \equiv \begin{cases} 5 \pmod{15} & \text{if } n \text{ is odd} \\ 10 \pmod{15} & \text{if } n \text{ is even.} \end{cases}$$

This allows us to give that

$$\begin{aligned} \#\{i \leq \varphi(5^n) : r_i \equiv 0 \pmod{3}\} &= \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 1 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 3 & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(5^n) : r_i \equiv 1 \pmod{3}\} &= \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 2 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 3 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$\#\{i \leq \varphi(5^n) : r_i \equiv 2 \pmod{3}\} = \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 1 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 2 & \text{if } n \text{ is even.} \end{cases}$$

Applying Lemma 2.3 to the preceding equalities gives that

$$\begin{aligned} \#\{i \leq \varphi(5^n) : q_i = 0\} &= \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 1 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 3 & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(5^n) : q_i = 1\} &= \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 2 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 2 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$\#\{i \leq \varphi(5^n) : q_i = 2\} = \begin{cases} 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 1 & \text{if } n \text{ is odd} \\ 4 \cdot \lfloor \frac{5^n}{15} \rfloor + 3 & \text{if } n \text{ is even.} \end{cases}$$

Since $1 + \omega + \omega^2 = 0$, we thus have that

$$\begin{aligned} \sum_{k=2+5^{n+1}}^{2+5^{n+1}+4 \cdot 5^n} \omega^{a_k} &= \sum_{j=0}^2 \#\{i \leq \varphi(5^{n+1}) : q_i = j\} \cdot \omega^j \\ &= \begin{cases} \omega & \text{if } n+1 \text{ is odd} \\ -\omega & \text{if } n+1 \text{ is even} \end{cases} \\ &= (-1)^n \omega, \end{aligned}$$

which proves part (i).

Part (ii) follows directly from Proposition 3.1 with $b = 3$ and $p = 5$. \square

Proof of Conjecture 1.1. Note that

$$\frac{1}{3^n 2^{3^n}} = \frac{8}{3^n} \cdot \frac{1}{4^{\frac{3}{2}(3^{n-1}+1)}}.$$

Let r_i and q_i be given by the base 4 algorithm for $8/3^n$. We will use the fact that each of these r_i are equivalent to 2 modulo 3. This is easily seen as we have for each i that $r_{i-1}4 = q_i 3^n + r_i$, so that taking this equality modulo 3 we have that $r_{i-1} \equiv r_i \pmod{3}$. Recalling that $r_0 = 8$ shows that indeed $r_i \equiv 2 \pmod{3}$ for each i .

Since $\text{ord}_{3^n} 4 = 3^{n-1}$, by Proposition 3.3, we have that

$$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} (e^{\frac{\pi i}{2}})^{a_k} = \sum_{j=0}^3 \#\{i \leq \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\frac{\pi i}{2}})^j.$$

Now $\#\{i \leq 3^n : q_i = j\}$ can be given by looking at where the number 3^n lies modulo 12. Since for every 12 consecutive numbers 4 of them are equivalent to 2 modulo 3 and these 4 fall into the 4 distinct equivalence classes modulo 4, we must consider the remainder of 3^n modulo 12. We have that

$$3^n \equiv \begin{cases} 3 \pmod{12} & \text{if } n \text{ is odd} \\ 9 \pmod{12} & \text{if } n \text{ is even.} \end{cases}$$

Thus we have that

$$\begin{aligned} \#\{i \leq \varphi(3^n)/2 : r_i \equiv 0 \pmod{4}\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(3^n)/2 : r_i \equiv 1 \pmod{4}\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(3^n)/2 : r_i \equiv 2 \pmod{4}\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$\#\{i \leq \varphi(3^n)/2 : r_i \equiv 3 \pmod{4}\} = \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is even.} \end{cases}$$

By Lemma 2.3, we have that

$$\begin{aligned} \#\{i \leq \varphi(3^n)/2 : q_i = 0\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(3^n)/2 : q_i = 1\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is even,} \end{cases} \\ \#\{i \leq \varphi(3^n)/2 : q_i = 2\} &= \begin{cases} \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$\#\{i \leq \varphi(3^n)/2 : q_i = 3\} = \begin{cases} \lfloor \frac{3^n}{12} \rfloor & \text{if } n \text{ is odd} \\ \lfloor \frac{3^n}{12} \rfloor + 1 & \text{if } n \text{ is even.} \end{cases}$$

Since $1 + (e^{\frac{\pi i}{2}}) + (e^{\frac{\pi i}{2}})^2 + (e^{\frac{\pi i}{2}})^3 = 0$, we thus have that

$$\begin{aligned} \sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} (e^{\frac{\pi i}{2}})^{a_k} &= \sum_{j=0}^3 \#\{i \leq \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\frac{\pi i}{2}})^j \\ &= \begin{cases} -1 & \text{if } n+1 \text{ is odd} \\ -i & \text{if } n+1 \text{ is even} \end{cases} \\ &= - \begin{cases} i & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

which proves part (i).

Part (ii) follows directly from Proposition 3.3 with $b = 2$ and $p = 3$. \square

Remark 4.1. The above two proofs can be easily modified to give similar results for any $\alpha_{b,p}$. The only thing that must be studied and applied is the distribution modulo b of the resulting sequence $r_1, r_2, \dots, r_{\text{ord}_p m b}$.

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